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Scaling Exponents for Active Scalars

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We provide bounds for Dirichlet quotients and for generalized structure functions for 3D active scalars and Navier–Stokes equations. These bounds put constraints on the possible extent of anomalous scaling.

KEY WORDS: Dirichlet quotients; 3D active scalars; Navier–Stokes equations.

1. INTRODUCTION

A central idea of classical[1] and modern[2] turbulence theories is scaling: certain averages of the hydrodynamical variables behave like powers of the independent variable. The basic hydrodynamic variable is the velocity $u(x, t)$, a three component divergence-free vector depending on position x and time t. The velocity obeys the incompressible Navier–Stokes equations, a nonlocal, nonlinear system of partial differential equations. The quantities that are asserted to possess scaling properties are obtained by taking averages of functionals of the velocity increments

$$\delta_y u(x, t) = u(x - y, t) - u(x, t)$$

for fixed y. Typical examples are the structure functions

$$\langle |\delta_y u|^m \rangle$$

for various powers of m. How to perform the proper average $\langle \cdots \rangle$ is not obvious. It would be desirable of course to have a specific and relatively simple procedure like time or space-time average. The very emergence of power laws from the nonlinear equations has not yet been demonstrated analytically, simple procedures or not.

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The traditional assumptions of scaling are of the form

$$\langle |\delta_y u|^m \rangle \sim U^m \left(\frac{|y|}{L}\right)^{\zeta_m}$$

where the constant U is a typical velocity magnitude, L a length magnitude and the scaling relation holds asymptotically in the limit of large Reynolds number

$$\mathscr{R} = \frac{UL}{\nu} \to \infty$$

Here ν is the kinematic viscosity. Moreover, scaling is expected to hold for $|y|$ in an inertial range that extends from a small cut-off length η to L. The cut-off length is determined by the kinematic viscosity and the average dissipation rate of kinetic energy ε. The exponents ζ_m are proportional to m in a Kolmogorov[1] theory:

$$\zeta_m = \frac{m}{3}$$

In recent years departures from this picture have been proposed in a variety of models (references in Ref. 2). Anomalous scaling—scaling that does not follow simple dimensional analysis—has been associated with intermittency, a term that usually is understood to refer to the uneven distribution in physical space of regions of high spatial gradients of the velocity. Temporal intermittency (uneven distribution in time of high temporal gradients) may also be present in these systems, and temporal and spatial intermittency could have distinct effects. This is the main theme of this paper.

The paper is divided in two parts. The first part (Sections 2 and 3) is introductory and is intended to serve as a background for the subsequent discussion; in it we will recall briefly the rapid change passive scalar model of R. Kraichnan[3] and his equation for structure functions. The equation is derived in a rational and formal but still non-rigorous manner. Kraichnan[4] made an ansatz regarding the dissipation that enabled him to calculate the exponents. The ansatz can be formulated as an assumption about certain Dirichlet quotients. One consequence of the ansatz is

$$\zeta_m \sim \sqrt{m}$$

for large m. To my knowledge, this very severe anomalous scaling has neither been proved nor disproved for the rapid change passive scalar.

In the second part of the paper we describe active scalar models and fractional structure functions. Active scalars[5] are natural incompressible hydrodynamical models of the full Navier–Stokes equations.[6, 7] Fractional structure functions are averages that generalize the usual structure functions: their spatial and temporal homogeneities can be different. We derive (rigorously) inequalities for appropriate Dirichlet quotients and for fractional structure functions both for the three dimensional Navier–Stokes equations and for the active scalar models. These inequalities impose limits on the possible anomalies and imply that the asymptotic behavior suggested by the ansatz of Kraichnan would require a specific and significant dominance of temporal intermittency over spatial intermittency. We conjecture that this is impossible in these deterministic and time correlated systems; more specifically, we believe that active scalars and regular Navier–Stokes dynamics obey

$$\lim_{m \to \infty} \inf \frac{\zeta_m}{m^s} = \infty$$

for every $0 \leqslant s < 1$.

A main feature both of our analysis and of the traditional assumptions regarding structure functions is that scaling extends down to a dissipation scale. That is where the exponents are anchored, and that's where we catch them.

2. PASSIVE SCALARS AND THE DOUBLE DUHAMEL FORMULA

A much studied,[8–13] model proposed by R. Kraichnan[3] is the random partial differential equation

$$(\partial_t + u \cdot \nabla - \kappa \Delta)\, \theta = f$$

where the incompressible velocity u is random, with given statistics. In this case scaling of velocity is an input, imposed from the outset. The problem is to determine the output—scaling exponents of the scalar structure functions

$$\langle |\delta_y \theta|^m \rangle$$

where the meaning of $\langle \cdots \rangle$ is expectation with respect to velocity statistics.

If the realizations of u are almost surely continuous in time, then one could reformulate the random partial differential equation as

$$\theta(x, t) = S_f(t) + S_u(t)$$

where

$$S_f(t) = e^{\kappa t \Delta}\theta_0 + \int_0^t e^{\kappa(t-s)\Delta} f(\cdot, s)\, ds$$

and

$$S_u(t) = -\int_0^t e^{\kappa(t-s)\Delta}(u(\cdot, s) \cdot \nabla\theta(\cdot, s))\, ds$$

This is an integral equation, the (single) Duhamel formula, used to define mild solutions in the deterministic PDE case. Mild solutions are a useful concept that generalizes the notion of solutions. But if the velocity is too singular in time then the single Duhamel formula does not suffice to make sense of solutions or to compute the higher order structure functions. This is the case if we assume that the velocities obey

$$\langle u_i(x, t)\, u_j(x-y, t+s)\rangle\, ds = DC_{ij}(y)\, \delta$$

The equality holds when left and right hand side are interpreted as measures in the variable s representing time; δ is Dirac measure concentrated at 0. The positive eddy diffusivity constant D is typically much larger than κ. The correlation tensor C_{ij} is non-dimensional. An example is given by

$$C_{ij}(y) \sim a\delta_{ij} - b\left(\frac{|y|}{\ell}\right)^{\zeta(u)}\left(\delta_{ij} - \frac{\zeta(u)}{\zeta(u)+d-1}\, \hat{y}_i \hat{y}_j\right)$$

The numbers a, b are non-dimensional and non-negative. The quantity ℓ has dimensions of length: it fixes length units. We denote

$$K_{ij}(y) = 2(C_{ij}(0) - C_{ij}(y))$$

The equation for the increment

$$(\delta_y\theta)(x) = \theta(x-y) - \theta(x)$$

is

$$(\partial_t - \kappa \Delta_x)\, \delta_y \theta + \partial_{x_j}(u_j\, \delta_y \theta) - \partial_{y_j}(\delta_y u_j\, \delta_y \theta) = \delta_y f$$

We denote for simplicity $q = \delta_y \theta$, $g = \delta_y f$.

One can write the equation for q as an iterated integral equation, the Double Duhamel Formula:

$$q(x, t; y) = \sum_{j=1}^{4} I_j(x, t; y) + \sum_{j=1}^{2} J_j(x, t; y) + S_g(x, t; y)$$

where the first four terms represent (u, u) interactions $((y, y), (y, x), (x, y)$ and $(x, x))$, the next two (u, g) interactions and the last one the pure g (and initial data) term. More generally, one can take any function Φ of one real variable and multiply the equation for q by $\Phi'(\delta_y \theta)$ to obtain

$$(\partial_t - \kappa \Delta_x)\, \Phi + \kappa \Phi'' \, |\nabla \delta_y \theta|^2 + \partial_{x_j}(u_j \Phi) - \partial_{y_j}(\delta_y u_j \Phi) = \Phi' \delta_y f$$

One can write a DDF for it: first one writes, using (single) Duhamel

$$\Phi(x, t; y) = I(x, t; y) + J(x, t; y)$$

where

$$\begin{aligned} I(x, t; y) = \int_0^t e^{\kappa(t-s)\Delta} [& \partial_{y_j}(\delta_y u_j(\cdot, s)\, \Phi(\cdot, s; y)) \\ & - \partial_{x_j}(u_j(\cdot, s)\, \Phi(\cdot, s; y))]\, ds \end{aligned}$$

and $J(x, t; y)$ involves all the rest. In the expression for I write again $\Phi(x, s; y) = I(x, s; y) + J(x, s; y)$, change the order of integration in time and keep spatial derivatives one on the outside and the other all the way inside. This leads to DDF for Φ. One of the relevant terms looks like this

$$I_1 = \int_0^t \left[\int_\sigma^t e^{\kappa(t-s)\Delta}\, \partial_{y_j}(\delta_y u_j(\cdot, s)\, e^{\kappa(s-\sigma)\Delta}\, \delta_y u_k(\cdot, \sigma)\, \partial_{y_k} ds \right] \Phi(\cdot, \sigma; y))\, d\sigma$$

One (formally) takes averages with respect to velocity statistics and one uses the following fundamental rules:

(1) θ, and consequently Φ depend on the past of the velocity but not on its future. Thus $\theta(t)$ and $\Phi(t)$ are *independent* of $u(s)$ for $s < t$.

(2) Homogeneity: $\langle \cdots \rangle$ commutes with spatial translation.

(3) The Stratonovich rule for stochastic integrals (midpoint in the Riemann sum discretization of the integral).

The main ingredients in the DDF are random linear operators with coefficients depending through $u(\cdot, t) \otimes u(\cdot, \sigma)$ on ordered times $t \geqslant \sigma$. They are applied to $\Phi(q)$ terms that depend on earlier times. The Stratonovich rule allows one to factor the expectation in half the product of the expectations. It is here where non-uniqueness of the model can manifest itself.

The average of the piece of the DDF we wrote above becomes for instance

$$\frac{D}{2} \int_0^t e^{\kappa(t-\sigma)\Delta} \partial_{y_j} K_{jk}(y) \partial_{y_k} \langle \Phi(\cdot, \sigma; y) \rangle \, d\sigma$$

The other pieces that have velocity-velocity interactions vanish because of the assumed spatial homogeneity. Moreover, spatial homogeneity reduces the operator $e^{\kappa(t-\sigma)\Delta}$ to the identity. It follows that

$$\langle \Phi(\cdot, \cdot; y) \rangle$$

obeys Kraichnan's equation

$$\partial_t \langle \Phi(\cdot, t; y) \rangle - \frac{D}{2} \partial_{y_i} (K_{ij}(y) \partial_{y_j} \langle \Phi(\cdot, t; y) \rangle) + \kappa \langle \Phi'' \, |\nabla \delta_y \theta|^2 \rangle = \langle \delta_y f \Phi' \rangle \tag{1}$$

Note that

$$D K_{ij}(y) \, \delta = \langle \delta_y u_i(x, t+s) \, \delta_y u_j(x, t) \rangle \, ds$$

follows from definitions and shows that K is a non-negative matrix. We have taken both the initial data and the sources to be deterministic and smooth. Natural modifications are needed for the cases they are random. On radial functions the second operator in (1) is, for example

$$-b\ell^{-\zeta(u)} D \frac{d-1}{\zeta(u) + d - 1} [r^{1-d} \partial_r (r^{d-1+\zeta(u)} \partial_r)]$$

3. SEARCHING FOR EXPONENTS

Let us seek time independent solutions. Also let us take Φ even and assume $\langle \Phi' \rangle = 0$. Finally, let us assume radial solutions. Then we need to solve, for example

$$b\ell^{-\zeta(u)} D \frac{d-1}{d-1+\zeta(u)} [r^{1-d} \partial_r (r^{d-1+\zeta(u)} \partial_r \langle \Phi \rangle)] = \kappa \langle \Phi'' \, |\nabla \delta_y \theta|^2 \rangle$$

Note that if we take

$$\Phi(q) = q^{2m}$$

where m is an integer, then pointwise

$$\Phi'' \, |\nabla_x \delta_y \theta|^2 = 2\left(2 - \frac{1}{m}\right) |\nabla_x \Psi|^2$$

where

$$\Psi(q) = q^m$$

and of course $\delta_y \theta$ is substituted for q. Thus we need to solve

$$r^{1-d} \partial_r (r^{d-1+\zeta} \partial_r \langle \Psi^2 \rangle) = \gamma \langle |\nabla_x \Psi|^2 \rangle \tag{2}$$

where

$$\gamma = \frac{2 - 1/m}{b} \left(\frac{d-1+\zeta}{d-1}\right)\left(\frac{\kappa}{D}\right) \ell^{\zeta} \tag{3}$$

and we wrote ζ for $\zeta(u)$. Now $\langle \Psi^2 \rangle$ vanishes quadratically at $r = 0$ so we can integrate the equation (2):

$$\langle \Psi^2 \rangle(r) = \gamma \frac{r^{1-\zeta}}{d-2+\zeta} \int_0^r \langle |\nabla_x \Psi|^2 \rangle(\rho) \left[\left(\frac{\rho}{r}\right)^{1-\zeta} - \left(\frac{\rho}{r}\right)^{d-1}\right] d\rho \tag{4}$$

It is natural to introduce the Dirichlet quotient

$$D_m = \frac{\ell^2 \langle |\nabla_x \Psi|^2 \rangle}{\langle \Psi^2 \rangle} \tag{5}$$

Let us now make the assumption that $\langle \Psi^2 \rangle$ is homogeneous. That means that

$$\frac{\langle \Psi^2 \rangle(\rho)}{\langle \Psi^2 \rangle(r)} = h_{2m}\left(\frac{\rho}{r}\right)$$

the function h_{2m} is a power,

$$h_{2m}(\lambda) = \lambda^{\zeta_{2m}}$$

Dividing by $\langle \Psi^2 \rangle(r)$ we have

$$\frac{\kappa(d-1+\zeta)(2-1/m)}{D(d-2+\zeta)(d-1)\,b}\left(\frac{r}{\ell}\right)^{2-\zeta}\int_0^1 D_m(r\lambda)\,h_{2m}(\lambda)(\lambda^{1-\zeta}-\lambda^{d-1})\,d\lambda=1$$

This equality can hold for a range of r only if

$$D_m(r)=C_m\left(\frac{r}{\ell}\right)^{\zeta-2} \tag{6}$$

This provides quite remarkable insight already: (6) must hold in order to sustain *any* homogeneous (power law) solution. The non-dimensional constant C_m may depend on m, d, ζ and on κ/D. Let us assume that h_{2m} is as above. Then we deduce

$$\zeta_{2m}(\zeta_{2m}+\zeta+d-2)=\left(\frac{d-1+\zeta}{d-1}\right)\left(\frac{2-1/m}{b}\right)\left(\frac{\kappa}{D}\right)C_m \tag{7}$$

So, in view of these considerations, a direct approach (numerical or analytical) is to study the independence of r and dependence of m of

$$C_m=\left(\frac{r}{\ell}\right)^{2-\zeta}\frac{\ell^2\langle|\nabla_x\Psi|^2\rangle}{\langle\Psi^2\rangle}$$

The anomalous results would follow from a bound that grows with m at a rate that is below m^2.

Kraichnan derived the equation already in the work.[3] We presented the DDF formulation because it is the natural integral equation formulation one would use in the case of rapidly oscillating deterministic coefficients. It might be also be useful for different equations. We write the dissipation terms of the Dirichlet quotient: that is where the puzzle is. Not only did Kraichnan derive the equation but he also went further and made a specific ansatz regarding the dissipation[4] that implies an asymptotic behavior of C_m for large m of the type $C_m \sim m$. This would imply

$$\zeta_m \sim \sqrt{m}$$

for large m. This, to my knowledge, is still an open problem despite vigorous research.

4. ACTIVE SCALARS

The dissipative active scalars we are considering here are solutions $\theta(x, t)$ of equations

$$(\partial_t + u \cdot \nabla - \nu \Delta)\, \theta = f \tag{8}$$

where $x \in \mathbf{R}^d$ represents position and $t \geq 0$ is time. The velocity u is divergence-free,

$$\nabla \cdot u = 0$$

and is determined by the scalar by a linear equation of state, for instance:

$$u_i(x, t) = \int \mathscr{K}_i(x - y)\, \theta(y, t)\, dy$$

or

$$\hat{u}_i(\xi, t) = \hat{\mathscr{K}}_i(\xi)\, \hat{\theta}(\xi, t)$$

The vector valued kernel $\mathscr{K}$ must be divergence-free. We will assume that the kernel is smooth away from the origin and that at the origin it may have at most a power-law singularity of order $\sigma \leq d$. In the case $\sigma = d$ we assume that the kernel is of classical Calderon–Zygmund type.[14] The simplest example of such an active scalar is the two dimensional quasi-geostrophic one that has merits of its own.[6, 7] In this paper we will not restrict ourselves to two dimensions, rather we would like to emphasize the three dimensional case. A very natural generalization of the two dimensional quasigeostrophic active scalar is given by the equation of state

$$u_i(x) = \varepsilon_{ijk} \frac{\partial h}{\partial x_j} R_k(\theta)$$

where ε_{ijk} is the totally anti-symmetric tensor (signum of the permutation $(1, 2, 3) \mapsto (i, j, k)$),

$$R_k = \frac{\partial}{\partial x_k} (-\Delta)^{-1/2}$$

are the Riesz transforms and h is any given smooth function. The usual two dimensional quasigeostrophic model is obtained for $h(x, y, z) = z$. The consideration below apply to all models in which θ and u are of the same order of magnitude (theta has dimensions of velocity). In view of the fact that these are scalar equations the maximum principle

$$\sup_x |\theta(x, t)| \leqslant \sup_x |\theta(x, 0)| + \int_0^t \sup_x |f(x, s)| \, ds$$

holds. This is a crucial advantage over (or departure from!) the Navier–Stokes equations. The energy equation also holds

$$\frac{1}{2} \frac{d}{dt} \int |\theta(x, t)|^2 \, dx + \nu \int |\nabla\theta(x, t)|^2 \, dx = \int f(x, t) \, \theta(x, t) \, dx$$

In view of the identity

$$\int u(x, t) \cdot \nabla\theta(x, t) \, \Delta\theta(x, t) \, dx = \sum_{j, k} \int \frac{\partial u_j}{\partial x_k}(x, t) \frac{\partial^2 \theta}{\partial x_j \, \partial x_k}(x, t) \, \theta(x, t) \, dx$$

straightforward use of the bound afforded by the maximum principle and the energy equality, it follows that the "enstrophy"

$$\int |\nabla\theta(x, t)|^2 \, dx$$

is bounded in time. This is enough for our purpose, but it is also sufficient in three spatial dimensions to show that the active scalar models have smooth solutions for all time. This is why we chose them, in addition to the Navier–Stokes equations. The question of regularity of solutions of the three dimensional Navier–Stokes equations being open we wanted to present our results not only for it but also for systems that we can control analytically.

5. STRUCTURE FUNCTIONS

Before we continue we need to define the averaging procedures that will replace taking the expectation with respect to velocity statistics. We will take averaging procedures we have used before,[15] namely long time averages and local averages on balls of size ρ:

$$M_\rho(h) = \langle h(x, t) \rangle = \sup_{x_0 \in R^d} \frac{1}{|B_\rho|} \limsup_{T \to \infty} \frac{1}{T} \int_{B_\rho} \int_0^T h(x, t) \, dt \, dx$$

where h is some function, B_ρ is the ball in R^d centered at x_0 and of radius ρ and $|B_\rho|$ is its volume. If the systems are taken to be periodic in space then a long time average followed by space average on the period box works as well. For a function h we define

$$s_m(h)(y) = s_m^{(\rho)}(h)(y) = \langle |\delta_y(h)|^m \rangle$$

where $(\delta_y(h))(x) = h(x-y) - h(x)$. This is the same as in Ref. 15 except that here we conform to the physics usage and do not take the mth root. The active scalar equation and the Navier–Stokes equations are of course different, but both are incompressible and non-local: in the Navier–Stokes equation the non-local effect is carried by the pressure, the active scalars have the non-local effect in the velocity. Both the pressure in the Navier–Stokes equation and the velocity in the active scalars are obtained by applying Calderon–Zygmund operators to the basic variable. The average defined above behaves well with respect to Calderon–Zygmund operators. We recall from Ref. 15 that, if K is a Calderon–Zygmund operator then

Proposition 1. The mth structure function of Kh satisfies

$$s_m(Kh)(y) \leqslant C_{a,b}(s_m(h))(y)$$

with $C_{a,b}(s_m(h))(y)$ defined by

$$C_{a,b}(s_m(h))(y) = C_{a,m} \int_{|\xi| \leqslant 2} (s_m(h)(|y|\,\xi))\, a^{-m}(|\xi|) \frac{d\xi}{|\xi|^d}$$

$$+ D_{b,m} \int_{|\xi| \geqslant 1} (s_m(h)(|y|\,\xi))\, b^{-m}(|\xi|) \frac{d\xi}{|\xi|^{d+1}}$$

where the weight functions a and b are arbitrary and the dependence of $C_{a,m}$, $D_{a,m}$ on them is given by

$$C_{a,m} = C^m \left(\int_0^2 a^q(r) \frac{dr}{r} \right)^{m/q}$$

and

$$D_{b,m} = C^m \left(\int_1^\infty b^q(r) \frac{dr}{r^2} \right)^{m/q}$$

with C depending on the kernel of the operator K.

Now $C_{a,b}(s_m(h))(y)$ does not differ from $s_m(h)(y)$ in what concerns the scaling exponent: if

$$s_m(h)(y) \leqslant H\left(\frac{|y|}{L}\right)^{\zeta_m}$$

holds with a constant H and all y then for every m there exist appropriate weights a, b and a constant γ_m depending on m such that

$$s_m(Kh)(y) \leqslant C_{a,b}(s_m(h))(y) \leqslant \gamma_m H\left(\frac{|y|}{L}\right)^{\zeta_m}$$

also holds for all y.

A few comments are needed. First, the dependence of γ_m on m is important. Inspection of the proposition above (see also Ref. 15) mindful of the change in notation) shows that

$$\gamma_m \leqslant C^m$$

with a fixed C. For the interested reader we note that the dependence of γ_m on m passes via ζ_m (our unknown!) but all we need is that $\zeta_m \leqslant Cm$ which is trivially true. Also we note that the classical result (see Ref. 14) on boundedness of Calderon–Zygmund operators in L^p spaces has worse dependence

$$\|Kh\|_{L^m}^m \leqslant (Cm)^m \|h\|_{L^m}^m$$

but here we have more than just L^m information in the assumption on $s_m(h)$. Incidentally, a dependence of γ_m of the type $(Cm)^m$ would not change the conclusion of the paper. Secondly, the inequality for $s_m(h)$ is assumed to hold for *all* y not only in an inertial range. This is however acceptable: if y is smaller than the viscous cut-off (the mathematically significant part) then the physical assumption would be even more stringent, $\zeta_m = m$, better behavior. If y is larger than L then there is no additional assumption, because the functions h we are going to consider are bounded.

Consider the finite difference operator δ_y, $(\delta_y \theta)(x, t) = \theta(x-y, t) - \theta(x, t)$. It follows from the active scalar equation that

$$(\partial_t + u\cdot\nabla - \nu\Delta)(\delta_y \theta) = (\delta_y f) + \partial_{y_j}((\delta_y u_j)(\delta_y \theta)) \tag{9}$$

In the case of the Navier–Stokes equations the corresponding equation is

$$(\partial_t + u\cdot\nabla - \nu\Delta)(\delta_y u_i) = (\delta_y f_i) + \partial_{y_j}((\delta_y u_j)(\delta_y u_i)) - \partial_{x_i}(\delta_y p) \qquad (10)$$

where p is the pressure. We will consider both cases simultaneously. We denote as before by $q = q(x, t; y)$ the increments. In the Navier–Stokes case q is a vector. Let us denote, also as before $\Psi = |q|^m$ and multiplying the evolution equation of q by $2mq_i\,|q|^{2m-2}$ deduce

$$(\partial_t + u\cdot\nabla - \nu\Delta)\,\Psi^2 + \frac{\nu}{4}\left(1 - \frac{1}{m}\right)|\nabla\Psi|^2 \leqslant \partial_{y_j}((\delta_y u_j)\,\Psi^2) + 2mq\cdot g\,|q|^{2m-2} \qquad (11)$$

where

$$g_i = \delta_y f - \partial_{x_i}(\delta_y p)$$

in the Navier–Stokes case and equals $\delta_y f$ for the active scalar. The first term in the right hand side of (11) looks the same as in the passive scalar case, but of course it is significantly different. In this case $\delta_y u(x, t)$ and $q(x, t-0; y)$ are not independent: they are identical in the Navier–Stokes equation case and linearly related in the active scalar one.

6. THE HEART OF THE MATTER

We will rearrange (11)

$$\frac{\nu}{4}\left(1 - \frac{1}{m}\right)|\nabla\Psi|^2 \leqslant I + II + III \qquad (12)$$

where the first term on the right hand side of (12) is

$$I = \partial_{y_j}((\delta_y u_j)\,\Psi^2)$$

the second is

$$II = 2mq\cdot g\,|q|^{2m-2}$$

(with $g_i = \delta_y f - \partial_{x_i}(\delta_y p)$ and respectively $g = \delta_y f$) and the third is

$$III = -(\partial_t + u\cdot\nabla - \nu\Delta)\,\Psi^2$$

We will study the averages of the terms in the right hand side of (12). We will keep track of the nonlocal terms and dependence of constants on m: it is important to note that the nonlocal terms enter only linearly in the balance equation. We start with I and observe that

$$I = \partial_{y_j}(\delta_y u_j \Psi^2) = -2m\, |q(x, t; y)|^{2m-2}\, (\delta_y u_j(x, t))\, q(x, t; y) \cdot \partial_{x_j} u(x - y)$$

in the Navier Stokes case and

$$I = \partial_{y_j}(\delta_y u_j \Psi^2) = -2m\, |q(x, t; y)|^{2m-2}\, (\delta_y u_j(x, t))\, q(x, t; y)\, \partial_{x_j} \theta(x - y)$$

for the active scalar. We take the average of I in the form above; we use first a Hölder inequality raising the term $|q|^{2m-1}$ to the $4m/2m-1$ power, the term $|\delta_y u|$ to the power $4m$ and the gradient term ($|\nabla u|$ or, respectively $|\nabla \theta|$) to the second power. We will use the same notation ε for the average energy dissipation rate

$$\varepsilon = \nu \langle |\nabla u|^2 \rangle$$

and

$$\varepsilon = \nu \langle |\nabla \theta|^2 \rangle$$

We obtain, in the case of the Navier–Stokes equations,

$$\langle |I| \rangle \leqslant 2m \sqrt{\frac{\varepsilon}{\nu}}\, (s_{4m}(u))^{1/2} \tag{13}$$

and in the active scalar case

$$\langle |I| \rangle \leqslant 2m \sqrt{\frac{\varepsilon}{\nu}}\, (s_{4m}(u))^{1/4m}\, (s_{4m}(\theta))^{2m-1/4m}$$

In the active scalar case we use the equation of state for velocity and Proposition 1 to conclude that

$$\langle |I| \rangle \leqslant 2m \sqrt{\frac{\varepsilon}{\nu}}\, (C_{a,b}(s_{4m}(\theta)))^{1/4m}\, (s_{4m}(\theta))^{2m-1/4m} \tag{14}$$

We pass now to the estimate of the average of the second term in the right hand side of (12),

$$II = 2mq \cdot g\, |q|^{2m-2}$$

In the case of the active scalar equation things are simple:

$$\langle |II| \rangle \leqslant 2m(s_{2m}(\theta))^{2m-1/2m}(s_{2m}(f))^{1/2m} \tag{15}$$

In the case of the Navier–Stokes equations we have the additional term involving the pressure:

$$II = 2mq \cdot \delta_y f\, |q|^{2m-2} + IV$$

with

$$IV = -2mq \cdot \nabla_x(\delta_y p)\, |q|^{2m-2}$$

The piece $2mq \cdot \delta_y f\, |q|^{2m-2}$ is estimate as II for the active scalar. We recall that the hydrodynamic pressure is

$$p = \sum_{j,k} R_j R_k(u_j u_k)$$

where $R_j = \partial_{x_j}(-\Delta)^{-1/2}$ are the Riesz transforms. Because the operation δ_y commutes with translation it follows that

$$\delta_y p = \sum_{j,k} R_j R_k(\delta_y(u_j u_k))$$

Now

$$\delta_y(u_j u_k) = (\delta_y u_j)\, u_k + (\tau_y u_j)(\delta_y u_k)$$

where $(\tau_y u_j)(x) = u_j(x-y)$. Therefore,

$$\delta_y p = \sum_{j,k} R_j R_k((\delta_y u_j)\, u_k + (\tau_y u_j)(\delta_y u_k))$$

Assuming that the Navier–Stokes velocity is bounded

$$\sup_x |u(x,t)| \leqslant U$$

then it follows[15] that

$$s_m(p)(y) \leqslant UC_{a,b}(s_m(u))(y) \tag{16}$$

and consequently, if

$$s_m(u)(y) \leqslant CU\left(\frac{|y|}{L}\right)^{\zeta_m}$$

holds for all y then

$$s_m(p)(y) \leqslant CU^2\gamma_m\left(\frac{|y|}{L}\right)^{\zeta_m}$$

also holds for all y, with the same dependence of $\gamma_m \leqslant C^m$ as above. The result about the pressure is a straightforward application of the result regarding the behavior of structure functions under Calderon–Zygmund operators. Before taking the average of IV we note that

$$IV = -2m\nabla\cdot(q(\delta_y p)\,|q|^{2m-2}) + V$$

with

$$V = 4(m-1)(\delta_y p)\,|q|^{m-2}\,q\cdot\nabla\Psi$$

we take the average of V and use a Holder inequality:

$$\langle V\rangle \leqslant 4(m-1)(s_{2m}(p))^{1/2m}\,(s_{2m}(u))^{m-1/2m}\,(\langle|\nabla\Psi|^2\rangle)^{1/2}$$

In view of the result above regarding the pressure we obtain

$$\langle V\rangle \leqslant 4(m-1)[\,CUC_{a,b}(s_{2m}(u))]^{1/2m}\,(s_{2m}(u))^{m-1/2m}\,(\langle|\nabla\Psi|^2\rangle)^{1/2} \quad (17)$$

We are left with terms that are in divergence form. If this was a periodic system or if we had an intersting homogeneous measure to average with, then these terms would have vanished after averaging. We want to record here the kind of errors they introduce. Thus, for instance it is easy to see, taking appropriate cut-off functions that

$$\langle III\rangle \leqslant C\left(\frac{U}{\rho}+\frac{\nu}{\rho^2}\right)s_{2m}^{(2\rho)}(u) \quad (18)$$

in the Navier–Stokes case and respectively

$$\langle III\rangle \leqslant C\left(\frac{U}{\rho}+\frac{\nu}{\rho^2}\right)s_{2m}^{(2\rho)}(\theta) \quad (19)$$

in the active scalar case. In the Navier–Stokes case we have an additional divergence term coming from the pressure and obtain the bound

$$-2m\langle \nabla\cdot(q(\delta_y p)\,|q|^{2m-2})\rangle \leqslant 2m\,\frac{C}{\rho}\,(s_{2m}^{(2\rho)}(u))^{2m-1/2m}\,(CUC_{a,b}(s_{2m}^{(2\rho)}))^{1/2m}$$

We are now ready to gather the bounds we obtained so far. We start with the active scalar. In view of (14), (15) and (19) we have a bound

$$\begin{aligned}\frac{\nu}{4}\left(1-\frac{1}{m}\right)\langle|\nabla\Psi|^2\rangle \leqslant{}& 2m\sqrt{\frac{\varepsilon}{\nu}}\,(C_{a,b}(s_{4m}(\theta)))^{1/4m}\,(s_{4m}(\theta))^{2m-1/4m}\\ &+2m(s_{2m}(\theta))^{2m-1/2m}\,(s_{2m}(f))^{1/2m}\\ &+C\left(\frac{U}{\rho}+\frac{\nu}{\rho^2}\right)s_{2m}^{(2\rho)}(\theta)\end{aligned}\tag{20}$$

In the Navier–Stokes case we have, in view of (13), (17), (18)

$$\begin{aligned}\frac{\nu}{4}\left(1-\frac{1}{m}\right)\langle|\nabla\Psi|^2\rangle \leqslant{}& 2m\sqrt{\frac{\varepsilon}{\nu}}\,(s_{4m}(u))^{1/2}\\ &+2m(s_{2m}(u))^{2m-1/2m}\,(s_{2m}(f))^{1/2m}\\ &+C\left(\frac{U}{\rho}+\frac{\nu}{\rho^2}\right)s_{2m}^{(2\rho)}(u)\\ &+2m\,\frac{C}{\rho}\,(s_{2m}^{(2\rho)}(u))^{2m-1/2m}\,(CUC_{a,b}(s_{2m}^{(2\rho)}(u)))^{1/2m}\\ &+4(m-1)[CUC_{a,b}(s_{2m}(u))]^{1/2m}\\ &\times(s_{2m}(u))^{m-1/2m}\,(\langle|\nabla\Psi|^2\rangle)^{1/2}\end{aligned}\tag{21}$$

The two inequalities (20) and (21) will be the basis of our main results. In order to simplify the exposition let us explain informally what they are accomplishing. First of all terms of the type $C_{a,b}(s_{jm}(h))$ with $j=2,4$ and $h=u$ or $h=\theta$ arize from the nonlocal contributions. If scaling is assumed then they are bounded by $C^{jm}s_{jm}(h)$. But these terms appear at powers $1/jm$, reflecting the fact that the nonlocal pieces in the balance equations are linear. Therefore the contribution of the prefactors C^{jm} is bounded as $m\to\infty$. (We could have afforded more, namely any algebraic growth in m.) Thus we use

$$(C_{a,b}s_m(h))^{1/m}\leqslant C(s_m(h))^{1/m}$$

for all $m \geqslant 2$, with $h=\theta$ or $h=u$. The degree of homogeneity in the unknown h of both the right and left hand sides in (20) and (21) is $2m$. The right hand side involves s_{4m} and s_{2m} with homogeneity one in s_{2m} and homogeneity 1/2 in s_{4m}. We shall see later that the left hand side can be brought to a form that involves $s_{jm}(h)$ for any j in $d=2$, and $s_{6m}(h)$ in $d=3$ and $s_{4m}(h)$ in $d=4$.

Let us do away with the last term in (21) at the expense of half the term on the left and write

$$\rho^2 \langle |\nabla \Psi|^2 \rangle \leqslant E(v, U, m, \rho, s_{2m}, s_{4m}) \tag{22}$$

In order to simplify the expressions E it is convenient to take ρ of the same order of magnitude as L and assume that $s_m^{(2\rho)}(v) \leqslant C s_m^{(\rho)}(v)$. Also we set

$$\mathscr{R} = \frac{UL}{v}$$

and use

$$\varepsilon \leqslant C \frac{U^3}{L}$$

which can be proven in both Navier–Stokes[15] and active scalar equations provided we take

$$\mathscr{P} = \frac{FL}{U^2}$$

bounded ($F = \|f\|_{L^\infty}$). We use also Young's inequality and obtain

$$E_\theta = C[m\mathscr{R}^{3/2}(s_{4m}(\theta))^{1/2} + m(\mathscr{R}+1)\, s_{2m}(\theta) + \mathscr{R} s_{2m}(f)] \tag{23}$$

and respectively

$$E_{NS} = C[m\mathscr{R}^{3/2}(s_{4m}(u))^{1/2} + (m^2\mathscr{R}^2 + m(\mathscr{R}+1))\, s_{2m}(u) + \mathscr{R} s_{2m}(f)] \tag{24}$$

It is interesting to note that the Navier–Stokes expression differs from the active sclar expression by one additional term: $m^2\mathscr{R}^2$ in the prefactor of s_{2m}. The inequality (22) with E defined above is the main result of this section. The bounds above can be viewed as bounds for the Dirichlet quotient

$$D_m(q) = \frac{\langle |\nabla \Psi|^2 \rangle}{\langle \Psi^2 \rangle}$$

of the type

$$D_m(q) \leqslant C[m\mathscr{R}^{3/2}F_{4m} + m\mathscr{R} + 1 + m\mathscr{R}G_{2m}] \tag{25}$$

and respectively

$$D_m(q) \leqslant C[m\mathscr{R}^{3/2}F_{4m} + m^2\mathscr{R}^2 + m\mathscr{R} + 1 + m\mathscr{R}G_{2m}] \tag{26}$$

where

$$F_{4m} = \frac{s_{4m}^{1/2}}{s_{2m}}$$

is a generalized flatness and

$$G_m = \frac{s_{2m}(f)}{s_{2m}}$$

is the ratio of the $2m$ structure functions of the force and the basic unknown.

7. FRACTIONAL STRUCTURE FUNCTIONS

We start with the local Sobolev inequality:

$$\left(\frac{1}{|B_\rho|}\int_{B_\rho} |\psi(x)|^{2^*}\,dx\right)^{2/2^*} \leqslant C\left[\rho^2 \frac{1}{|B_\rho|}\int_{B_\rho} |\nabla\psi(x)|^2\,dx + \frac{1}{|B_\rho|}\int_{B_\rho} |\psi(x)|^2\,dx\right]$$

where ψ is a function, the constant C depends only on the dimension d and

$$2^* = \frac{2d}{d-2}$$

Taking the time average of the local Sobolev inequality with $\psi = \Psi$ we obtain

$$\sup_{x_0} \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{|B_\rho|}\int_{B_\rho} |\Psi(x)|^{2^*}\,dx\right)^{2/2^*} dt \leqslant E$$

For arbitrary $m \geqslant 1$ and $0 \leqslant r \leqslant 1$ let us define fractional structure functions

$$s_{m;\,r}(h) = \sup_{x_0} \lim_{T \to \infty} \sup \frac{1}{T} \int_0^T \left(\frac{1}{|B_\rho|} \int_{B_\rho} |\delta_y h(x)|^m \, dx \right)^r dt$$

The usual structure function of order m is $s_{m;\,1}$ and clearly, if $r \leqslant 1$ we have only the fraction r of the usual expression. It is perhaps useful to observe that the function

$$(s_{m;\,r}(h))^{1/rm}$$

is a $L^{mr}(dt;\, L^m_{loc;\,unif}(dx))$ seminorm and a nondecreasing function of each of the arguments m and r.

The considerations above lead to bounds of $s_{2^*m;\,2/2^*}$ in terms of $s_{4m;\,1/2}$ and $s_{2m;\,1}$. The expression E can be easily replaced (easily means by taking the time average at the appropriate step in the derivation) by

$$E^* = C[m\mathscr{R}^{3/2} s_{4m;\,1/2} + (m\mathscr{R} + 1)\, s_{2m;\,1} + s_{2m;\,1}(f)]$$

for the active scalar and respectively

$$E^* = C[m\mathscr{R}^{3/2} s_{4m;\,1/2} + (m^2\mathscr{R}^2 + m\mathscr{R} + 1)\, s_{2m;\,1} + s_{2m;\,1}(f)]$$

for the Navier–Stokes equation. In $d = 3$, where $2^* = 6$ we obtained thus

$$s_{6m;\,1/3} \leqslant E^* \tag{27}$$

This is the main result of this section.

8. SCALING EXPONENTS

We will make assumptions regarding the nature of the functions $s_{m;\,r}$ and deduce the relevant conclusions from the inequalities above. Let us assume thus that

$$(s_{m;\,r}(v))(y) \sim \left(\frac{|y|}{L} \right)^{\zeta_{m;\,r}}$$

holds for

$$\eta \leqslant |y| \leqslant L$$

where

$$\eta = \nu^{3/4}\varepsilon^{-1/4}$$

is the Kolmogorov dissipation length. In view of the remark in the previous section, if $\eta/L \to 0$ then, from $m_1 \leqslant m_2$ and $r_1 \leqslant r_2$ the convexity inequalities

$$\frac{\zeta_{m_2;\, r_2}}{m_2 r_2} \leqslant \frac{\zeta_{m_1;\, r_1}}{m_1 r_1}$$

follow. Also, it is easy to check directly using the Hölder inequality that the monotonicity inequalities

$$\zeta_{m;\, r} \leqslant \zeta_{mr;\, 1}$$

hold.

Both left and right hand sides of (27) are homogeneous of degree $2m$ in the variable $\delta_y \theta$ respectively $\delta_y u$. We take the $1/2m$ root of both sides and use the scaling assumptions. We note that prefactors that are polynomials in m become bounded by absolute constants after this operation. The Reynolds number dependence will be expressed in terms of the Kolmogorov dissipation length. We obtain, in both the Navier–Stokes case and the active scalar one

$$\begin{aligned}\left(\frac{|y|}{L}\right)^{\zeta_{6m;\, 1/3}/2m} &\leqslant C\left(\frac{|y|}{L}\right)^{\zeta_{4m;\, 1/2}/2m}\left(\frac{\eta}{L}\right)^{-1/m}\\ &\quad + C\left[\left(\frac{|y|}{L}\right)^{\zeta_{2m;\, 1}/2m} + \left(\frac{|y|}{L}\right)^{\phi_{2m;\, 1}/2m}\right]\left(\frac{\eta}{L}\right)^{-2/3m} \qquad (28)\end{aligned}$$

where we denoted by $\phi_{2m;\, 1}$ the scaling exponents of the fractional structure functions associated to the forcing term f. We will evaluate (28) at the bottom of the scaling range $|y| \sim \eta$. As $\eta/L \to 0$ we deduce the necessary condition

$$\begin{aligned}\min\Bigg\{ -\frac{\zeta_{6m;\, 1/3}}{2m} + \frac{\zeta_{4m;\, 1/2}}{2m} - \frac{1}{m};\ -\frac{\zeta_{6m;\, 1/3}}{2m} + \frac{\zeta_{2m;\, 1}}{2m} - \frac{2}{3m};\\ -\frac{\zeta_{6m;\, 1/3}}{2m} + \frac{\zeta_{2m;\, 1}}{2m} - \frac{2}{3m}\Bigg\} \leqslant 0\end{aligned}$$

From the convexity inequalities and the mild requirement that

$$\phi_{2m;\, 1} \geqslant \zeta_{2m;\, 1}$$

it follows that

$$\frac{\zeta_{6m;\,1/3}}{2m} \geqslant \frac{\zeta_{4m;\,1/2}}{2m} - \frac{1}{m} \tag{29}$$

If the scaling ranges depend on m but there is no Reynolds number independent small scale cut-off then the same kind of inequality holds. A popular belief is that the $m-th$ structure function has a scaling range that extends down to the scale $\eta_{m;\,r}$ at which the local Reynolds number based on that particular structure function is of order one. In that case

$$\mathscr{R}\left(\frac{\eta_{m;\,r}}{L}\right)^{(1+\zeta_{m;\,r}/mr)} \sim 1$$

and the inequality becomes

$$\frac{\zeta_{6m;\,1/3}}{2m} \geqslant \frac{\zeta_{4m;\,1/2}}{2m} - \frac{3}{4m}\left(1 + \frac{\zeta_{6m;\,1/3}}{2m}\right)$$

9. INTERPRETATION

When one takes averages in space and time of nonlinear expressions, the order in which the operations are performed is of consequence.

$$\frac{1}{T}\int_0^T\left(\frac{1}{|B_\rho|}\int |\delta_y u|^m\,dx\right)^r dt \leqslant \left[\frac{1}{T}\int_0^T \frac{1}{|B_\rho|}\int |\delta_y u|^m\,dx\,dt\right]^r$$

is always true for $0 \leqslant r \leqslant 1$, but in general the two integrals are not equal. On solutions of partial differential equations these integrals might however scale asymptotically in the same way as functions of y. That would imply that

$$\zeta_{m;\,r} = r\zeta_m \tag{30}$$

while, in general, the convexity inequalities guarantee only

$$\zeta_{m;\,r} \geqslant r\zeta_m \tag{31}$$

If the equality (30) would hold then (29) would become

$$\frac{\zeta_{6m}}{6m} \geqslant \frac{\zeta_{4m}}{4m} - \frac{1}{m}$$

and this would be inconsistent with

$$\zeta_m = K\sqrt{m} + O(m^p), \qquad p < \tfrac{1}{2} \tag{32}$$

One might argue that the equality assumption (30) seems reasonable only if there is a uniformly smooth behavior in time. But more is true. If we use both the monotonicity and convexity inequalities we get in general that

$$r\zeta_m \leqslant \zeta_{m;\,r} \leqslant \zeta_{mr} \tag{33}$$

In addition, from spatial interpolation we deduce that, if

$$m_1 r_1 = m_2 r_2 = m_3 r_3$$

and

$$m_1 \leqslant m_2 \leqslant m_3$$

then

$$\zeta_{m_2;\,r_2} \geqslant (1-a)\,\zeta_{m_1;\,r_1} + a\zeta_{m_3;\,r_3}$$

holds with

$$a = \frac{m_3(m_2 - m_1)}{m_2(m_3 - m_1)}$$

In particular if $(m_1, m_2, m_3) = (2m, 4m, 6m)$ and $(r_1, r_2, r_3) = (1, \tfrac{1}{2}, \tfrac{1}{3})$ then

$$\zeta_{4m;\,1/2} \geqslant \tfrac{1}{4}\zeta_{6m;\,1/3} + \tfrac{3}{4}\zeta_{2m;\,1}$$

Therefore (29) implies that

$$\zeta_{6m;\,1/3} \geqslant \zeta_{2m} - \tfrac{8}{3} \tag{34}$$

holds. In view of (33) it follows that

$$\zeta_{m;\,r} = (1 - a_{m;\,r})\,\zeta_{mr} + a_{m;\,r} r\zeta_m \tag{35}$$

holds with $0 \leqslant a_{m;\,r} \leqslant 1$. Then it follows from (34) that

$$\frac{\zeta_{6m}}{3} - \zeta_{2m} \geqslant -\frac{8}{3a_{6m;\,1/3}} \tag{36}$$

This again is inconsistent with (32) unless

$$a_{6m;\,1/3} = O(m^{-1/2})$$

This is means that (32) is violated unless time singularities dominate in a significant and very particular way over spatial ones. I believe that further work will rule out this last possibility, at least for active scalars. Actually, I believe that, for any $0 \leqslant s < 1$,

$$\lim_{m \to \infty} \inf \frac{\zeta_m}{m^s} = \infty$$

is true for active scalars and regular solutions of Navier–Stokes equations.

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